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On the solution of the integro–differential fragmentation equation with continuous mass loss

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Abstract

In this paper we obtain a complete classification of all possible non-trivial similarity solutions of the integro–differential fragmentation equation with continuous mass loss rate. These solutions include the effects of both continuous and discrete mass loss rates. The similarity solutions are compared with the solutions found earlier. The comparison shows that some previous solutions are obtained as special cases from our solutions. The results reported here provide further evidence of the usefulness of the Lie group method for obtaining similarity solutions for either differential or integro–differential equations.

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1. Introduction

During the last two decades, the fragmentation equation has attracted significant attention from a diverse group of scientists such as physicists and mathematicians [1–4]. This is because the fragmentation equation not only arises from realistic physical phenomena, but also can be widely applied to many physically significant problems including reacting polymers, clustering of colloidal particles, astrophysics and birth–death processes [5–7].

Consider the fragmentation equation with continuous mass loss case

$$\frac{\partial n(\tilde{x},t)}{\partial t} = -a(\tilde{x})n(\tilde{x},t) + \int_{\tilde{x}}^{\infty} a(y)K(\tilde{x},y)n(y,t)\,\mathrm{d}y + \frac{\partial}{\partial \tilde{x}}[c(\tilde{x})n(\tilde{x},t)]. \tag{1}$$

This equation describes the evolution of the particle mass distribution $n(\tilde{x},t)$ for a system of particles undergoing fragmentation with continuous mass loss rate $c(\tilde{x})$ where $a(\tilde{x})$ is the fragmentation rate, and $K(\tilde{x},y)$ is the distribution of daughter particles with mass $\tilde{x}$ spawned by the fragmentation of a parent of mass y.

Since continuous mass loss involves no collisions between particles and depends only on the interaction between each particle and its environment, the fragmentation equation (1) is a linear integro–differential equation.

Different choices of the coefficients in equation (1) will lead to different homogeneous fragmentation models. According to the work of Edwards *et al* [8], Cai *et al* [9] and Baumann *et al* [10], we consider the following power-law dependence:

$$a(\tilde{x}) = \tilde{x}^{\alpha} \qquad K(\tilde{x}, y) = 2\varphi \tilde{x}^{\upsilon} y^{-(\upsilon+1)} \qquad c(\tilde{x}) = \varepsilon \tilde{x}^{-\gamma} \qquad \varepsilon \geqslant 0. \tag{2}$$

Via the transformation

$$u(x, t) = \tilde{x}^{-\upsilon} n(\tilde{x}, t) \qquad x = \tilde{x}^{\varphi} \tag{3}$$

the partial integro–differential fragmentation (PIDF) equation (1) can be written in the form

$$\frac{\partial u}{\partial t} + (x^{n+1} - ax^m)u - bx^{m+1}\frac{\partial u}{\partial x} - 2\int_x^{\infty} x'^n u(x', t)\,\mathrm{d}x' = 0 \tag{4}$$

where

$$n = \frac{\alpha}{\varphi} - 1 \qquad m = \frac{\gamma - 1}{\varphi} \qquad a = b[1 + m + (\upsilon + \lambda(\upsilon + 2))/2\varphi] \qquad b = \varepsilon\varphi. \tag{5}$$

We have to mention that in the case of the absence of mass loss (sometimes called discrete mass loss), the coefficient $c(x)$ equals zero and then equations (2) and (5) show that $\varepsilon = b = a = 0$.

Hence, the fragmentation equation (1) is reduced to that considered by Baumann *et al* [10] in their work.

Differentiating the above equation (4) with respect to x, the integro–differential equation (4) is transformed to the following partial differential equation with variable coefficients:

$$\frac{\partial^2 u}{\partial t\partial x} + A\frac{\partial^2 u}{\partial x^2} + \beta\frac{\partial u}{\partial x} + Cu = 0 \tag{6}$$

where

$$\begin{aligned} A &= -bx^{m+1} \\ \beta &= -bx^m + x^{n+1} \\ C &= (n+3)x^n + ambx^{m-1}. \end{aligned} \tag{7}$$

Various methods for seeking explicit solutions to the fragmentation equation have been proposed among which are the Laplace transformation [11], moment expansion [12] and similarity method [10, 13]. In a work on similarity solution, Baumann *et al* [10] discussed the case of discrete mass loss ($\varepsilon = 0$) and obtained similarity solutions of the partial differential equation (6) instead of the PIDF equation (1).

In the case of continuous loss of mass ($\varepsilon \neq 0$), Saied and El-Wakil [13] extended the analysis of Baumann *et al* [10] and classified the similarity solutions of the partial differential fragmentation equation (6) in terms of Lie group parameters. In this work we aim to generalize the analysis of Saied and El-Wakil [13] and Baumann *et al* [10], by applying the Lie method to the integro–differential equation (4) rather than the corresponding differential equation (6).

The paper is arranged as follows. In section 2, we introduce Lie analysis for integro–differential equations. In section 3, we derive the similarity reductions. Section 4 is devoted to finding different classes of similarity solutions. A discussion and concluding remarks are presented in section 5.

2. Application of the Lie method

In this section, the Lie method [14, 15] will be applied to find the similarity solutions and similarity reductions of the PIDF equation (4). Now we consider the one-parameter (ε) group of infinitesimal transformations

$$\begin{aligned}
x^* &= x + \varepsilon\xi(x,t,u) + O(\varepsilon^2)\\
t^* &= t + \varepsilon T(x,t,u) + O(\varepsilon^2)\\
u^* &= u + \varepsilon\eta(x,t,u) + O(\varepsilon^2)\\
u^*_{x^*} &= u_x + \varepsilon[\eta_x] + O(\varepsilon^2)\\
u^*_{t^*} &= u_t + \varepsilon[\eta_t] + O(\varepsilon^2)\\
\int_{x^*}^{\infty} x'^n u^*\,\mathrm{d}x' &= \int_x^{\infty} x'^n u\,\mathrm{d}x' + \varepsilon[\Delta I]
\end{aligned} \tag{8}$$

where the infinitesimals $[\eta_x]$ and $[\eta_t]$ are given by

$$\begin{aligned}
[\eta_x] &= \eta_x + (\eta_u - \xi_x)u_x - T_x u_t - \xi_u u_x^2 - T_u u_x u_t\\
[\eta_t] &= \eta_t + (\eta_u - T_t)u_t - \xi_t u_x - T_u u_t^2 - \xi_u u_x u_t.
\end{aligned} \tag{9}$$

In addition, it can be proved that

$$[\Delta I] = -\xi x^n u \int_x^{\infty} \left[n\xi x'^{n-1} u + x'^n \eta + x'^n u \left[\frac{\partial\xi}{\partial x'} + \frac{\partial\xi}{\partial u}\frac{\partial u}{\partial x'} \right]\right] \mathrm{d}x'. \tag{10}$$

If a similarity reduction of equation (4) is to be found by the Lie method, it is necessary to determine the functions $\eta(x, t, u)$, $T(x, t, u)$ and $\xi(x, t, u)$ that leave PIDF equation (4) invariant under the group of transformations (8). The infinitesimal criteria for the invariance of (4) under the group (8) are given by

$$\chi H = \lambda(x,t,u) H \tag{11}$$

where

$$\chi = \xi\frac{\partial}{\partial x} + T\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial u} + [\eta_x]\frac{\partial}{\partial u_x} + [\eta_t]\frac{\partial}{\partial u_t} + [\Delta I]\frac{\partial}{\partial\int x'^n u\,\mathrm{d}x'}. \tag{12}$$

$$H\left(x, t, u, u_x, u_t + \int_x^{\infty} x^n u\,\mathrm{d}x\right) = u_t + (x^{n+1} - ax^m)u - bx^{m+1}u_x - \int_x^{\infty} x'^n u\,\mathrm{d}x' \tag{13}$$

where $\lambda(x, t, u)$ is an arbitrary function to be determined. Therefore, substituting (12) and (13) into (11), we obtain the equation

$$\begin{aligned}
&[\eta_t] + (x^{n+1} - ax^m)\eta + \xi((n+1)x^n - amx^{m-1})u - b(m+1)\xi x^m u_x\\
&\quad - bx^{m+1}[\eta_x] - 2\int_x^{\infty}\left(n\xi x'^{n-1}u + x'^n\eta + x'^n u\left(\frac{\partial\xi}{\partial x} + \frac{\partial\xi}{\partial u}\frac{\partial u}{\partial x'}\right)\right)\mathrm{d}x'\\
&= \lambda(x,t,u)\left[u_t + (x^{n+1} - ax^m)u - bx^{m+1}u_x - 2\int_x^{\infty} x'^n u\,\mathrm{d}x'\right].
\end{aligned} \tag{14}$$

Now we substitute expressions (9) and (10) into equation (14). Setting the coefficients with like derivatives and integral to zero, we obtain the determining equations for T, ξ, η and λ. These equations are

$$\begin{aligned}
&\xi_u = T_u = T_x = 0\\
&\eta_u - T_t = \lambda(x, t, u)\\
&\int_x^\infty (n\xi x'^{n-1}u + x^n\eta + x'^n u\xi_x)\,\mathrm{d}x' = \lambda(x, t, u)\int_x^\infty x'^n u\,\mathrm{d}x' \\
&\xi_t + b(m+1)\xi x^m + bx^{m+1}(\eta_u - \xi x) = \lambda bx^{m+1}\\
&\eta_t + (x^{n+1} - ax^m)\eta + \xi((n+1)x^n - amx^{m-1})u - bx^{m+1}\eta_x = \lambda(x^{n+1} - ax^m)u.
\end{aligned} \tag{15}$$

Solving the whole system of the determining equation (15) in the case $m = n + 1$ yields, after lengthy manipulations, the following expressions for T, ξ, η:

$$\begin{aligned}
T &= \frac{a_0}{2}t^2 + a_1t + a_2\\
\xi &= -\frac{1}{m}(a_0t + a_1)x + f_0x^{-n}\\
\eta &= (mf_0(a-1)t + a_3)u
\end{aligned} \tag{16}$$

with four arbitrary constants a_0, a_1, a_2, a_3 since

$$a_0 = 2f_0m^2b. \tag{17}$$

It is to be noted that the parameter a_0 was completely missed in the work of Saied and El-Wakil and consequently the corresponding similarity solution and reduction were also missed. Knowledge of the infinitesimal elements T, ξ and η given in (16) enables us to construct four operators (for details see Bluman and Kumie [14] and Olver [15]),

$$\begin{aligned}
\chi_1 &= \frac{\partial}{\partial t}\\
\chi_2 &= t\frac{\partial}{\partial t} - \frac{1}{m}x\frac{\partial}{\partial x}\\
\chi_3 &= \frac{t^2}{2}\frac{\partial}{\partial t} + \left(-\frac{1}{m}tx + \frac{1}{2m^2b}x^{1-m}\right)\frac{\partial}{\partial x} + \frac{a-1}{2mb}tu\frac{\partial}{\partial u}\\
\chi_4 &= u\frac{\partial}{\partial u}.
\end{aligned} \tag{18}$$

It is obvious that the vector fields χ_2 and χ_4 contain the scaling properties of the fragmentation equation (4) and χ_1 represents a translation in time.

The commutation relations of the four vector fields χ_1, χ_2, χ_3 and χ_4 are shown as follows:

$$[\chi_1, \chi_2] = \chi_1 \qquad [\chi_1, \chi_3] = \chi_2 \qquad [\chi_2, \chi_3] = \chi_3 \tag{19}$$

and the rest equal zero.

We have to note that in the case of $m \neq n + 1$, the determining equations (15) show that the fragmentation (4) admits the trivial group $\xi = 0$, $T = a_2$, $\eta = a_3u$. This means that the fragmentation equation (4) is invariant under time translation and possesses a scaling invariance in the dependent variable u.

In the absence of the continuous mass loss where $\varepsilon = a = b = 0$ and equation (17) implies $a_0 = 0$, the group (16) becomes

$$T = a_1t + a_2 \qquad \xi = -\frac{1}{m}a_1x + f_0x^{1-m} \qquad \eta = (-mf_0t + a_3)u. \tag{20}$$

The corresponding vector fields in the case $\varepsilon = 0$ are given by

$$\chi_1 = \frac{\partial}{\partial t}$$
$$\chi_2 = u\frac{\partial}{\partial u}$$
$$\chi_3 = t\frac{\partial}{\partial t} - \frac{1}{m}x\frac{\partial}{\partial x}$$
$$\chi_4 = \frac{1}{m}x^{1-m}\frac{\partial}{\partial x} - tu\frac{\partial}{\partial u}. \qquad (21)$$

The vector fields χ_2 and χ_3 contain the properties of the scaling invariant nature of the fragmentation equation while χ_1 represents time translation invariance.

The commutation relations of the four vector fields χ_1, χ_2, χ_3 and χ_4 are shown as follows:

$$[\chi_1, \chi_3] = \chi_1 \qquad [\chi_1, \chi_4] = \chi_2 \qquad [\chi_3, \chi_4] = \chi_4 \qquad (22)$$

whereas the rest equal zero.

3. Similarity reductions

In order to obtain the similarity reductions for fragmentation equation (4), we have to solve first the characteristic equations

$$\frac{\mathrm{d}t}{T} = \frac{\mathrm{d}x}{\xi} = \frac{\mathrm{d}u}{\eta} \qquad (23)$$

associated with the vector fields χ_1, χ_2, χ_3 and χ_4 and their combinations. In general, the solution of (23) will involve two arbitrary constants, of which one constant plays the role of similarity variable s and the other, say $F(s)$, plays the role of similarity solution. Substituting the variables s and $F(s)$ into fragmentation equation (4) results in an ordinary integro–differential equation.

Because a linear combination of the four vector fields determines the general symmetry of equation (4), we can use a combination of the vector fields to classify the types of solutions. However, the similarity forms and reductions of the fragmentation equation (4) in the absence of mass loss, i.e. $\varepsilon = 0$, are listed in table 1 while the reduction corresponding to the case of mass loss, i.e. $\varepsilon \neq 0$, is tabulated in table 2.

4. Similarity solutions

In this section we shall discuss the similarity solution of the fragmentation equation (4) in two different cases: (i) $\varepsilon = 0$ and (ii) $\varepsilon \neq 0$.

(i) Case $\varepsilon = 0$. In spite of scaling, invariant solutions of the fragmentation equation hold great interest because of evidence [11, 17] that large classes of general solutions tend to scaling solutions after initial transients decay away. Nevertheless the nature of solutions in the non-scaling regime deserves to be obtained explicitly. Therefore, we focus our attention on obtaining the general similarity solution where the other classes listed in table 1 can be obtained as particular cases.

As shown in table 1, the general similarity solution is

$$u = F(s)(a_1 t + a_2)^k \exp\left[-\frac{mf_0}{a_1}t\right] \qquad (24)$$

Table 1. Similarity forms and reduced fragmentation equation in the case $\varepsilon = 0$.

Case	Similarity forms	Reduced fragmentation equation
1	$u = F(s)(a_1t + a_2)^k \exp\left[-\frac{mf_0}{a_1}t\right]$ $s = (a_1t + a_2)(mf_0 - a_1x^m)$ $m = \frac{1}{a_1^2}(a_2a_1 + a_1mf_0)$	$a_1s\frac{dF}{ds} + \left(a_1m - \frac{m}{a_1}s\right)F(s) + \frac{2}{a_1}\int_s^\infty F(s')\,ds' = 0$
2	$u = F(s)(a_1t + a_2)^{a_2/a_1}$ $s = x(a_1t + a_2)^{1/m}$	$\frac{a_1}{m}s\frac{dF}{ds} + (a_3 + s^m)F(s) - 2\int_s^\infty s'^{m-1}F(s')\,ds' = 0$
3	$u = F(s)\exp\left[\frac{a_3}{mf_0} - s\right]x^m$ $s = t$	$m\left(\frac{a_3}{mf_0} - s\right)\frac{dF}{ds} + 2F(s) = 0$
4	$u = F(s)\,e^{\frac{1}{a_2}(a_3t - \frac{mf_0}{2}t^2)}$ $s = \frac{mf_0}{a_2}t - x^m$	$\frac{mf_0}{a_2}\frac{dF}{ds} + \left(\frac{a_3}{a_2} - s\right)F(s) + \frac{2}{m}\int_s^\infty F(s')\,ds' = 0$
5	$u = F(s)$ $s = x$	$s^mF(s) - 2\int_s^\infty s'^{m-1}F(s')\,ds' = 0$
6	$u = F(s)\,e^{\frac{a_3}{a_2}t}$ $s = x$	$\left(\frac{a_3}{a_2} + s^m\right)F(s) - 2\int_s^\infty s'^{m-1}F(s')\,ds' = 0$

Table 2. Similarity forms and reduced fragmentation equation in the case $\varepsilon \neq 0$.

Case	Similarity forms	Reduced fragmentation equation
χ_1	$s = x$ $u = F(s)$	$(1-a)s^mF(s) - bs^{m+1}\frac{dF(s)}{ds} - 2\int_s^\infty s'^{m-1}F(s')\,ds' = 0$
χ_2	$s = tx^m$ $u = F(S)$	$s\frac{dF}{ds} + (1-a)sF(s) - mbs^2\frac{dF}{ds} - \frac{2}{m}\int_s^\infty F(s')\,ds' = 0$
χ_3	$s = t^2x^m - \frac{1}{mb}t$ $u = t^{\frac{a-1}{mb}}F(S)$	$(1-a) - sF(s) - \frac{1}{mb}s^2\frac{dF}{ds} - \frac{2}{M}\int_s^\infty F(s')\,ds' = 0$
$\chi_2 + \gamma\chi_4$	$s = tx^m$ $u = t^\gamma F(s)$	$(\gamma + (1-a)s)F(s) - s(1 - mbs)\frac{dF}{ds} - \frac{2}{M}\int_s^\infty F(s')\,ds' = 0$
$\chi_1 - \gamma\chi_4$	$s = x$ $u = F(s)\exp[-\gamma t]$	$(-\gamma + (1-a)s^m)F(s) - bs^{m+1}\frac{dF}{ds} - 2\int_s^\infty s'^{m-1}F(s')\,ds' = 0$
$\chi_1 + \chi_2$ $+ \chi_3 + \chi_4$	$s = \left(\frac{a_0}{2}t^2 + a_1t + a_2\right)x^m - mf_0t$ $u = F(s)\exp\int\frac{A(t)}{B(t)}\,dt$ $A(t) = mf_0(a-1)t + a_3$ $B(t) = \frac{a_0}{2}t^2 + a_1t + a_2$	$(s^2 - A_1s + A_2)\frac{dF}{ds} - (A_3s + A_4)F(s) + A_5\int_s^\infty F(s')\,ds' = 0$ $A_1 = \frac{a_1}{bm}, A_2 = \frac{a_0}{2b^2m^2}, A_3 = \frac{a_3}{bm}, A_4 = \frac{a_3}{mb}, A_5 = \frac{1}{bm^2}$

where

$$s = (a_1t + a_2)(mf_0 - a_1x^m) \qquad k = \frac{1}{a_1^2}(a_1a_3 + a_2mf_0). \tag{25}$$

Substitution of this similarity solution into (4) results in the following reduced fragmentation equation:

$$a_1s\frac{dF}{ds} + \left(a_1k - \frac{m}{a_1}s\right)F(s) + \frac{2}{a_1}\int_s^\infty F(s')\,ds' = 0. \tag{26}$$

The use of the substitution

$$g(s) = \int_s^\infty F(s')\,ds' \tag{27}$$

transforms equation (26) into the following second order differential equation with variable coefficients:

$$s\frac{\mathrm{d}^2g}{\mathrm{d}s^2}+\left(k-\frac{m}{a_1^2}s\right)\frac{\mathrm{d}g}{\mathrm{d}s}-\frac{2}{a_1^2}g(s)=0. \tag{28}$$

Rescaling s by $\zeta = \frac{m}{a_1^2}s$, equation (28) can be reduced to the standard form of Kummer's equation

$$\zeta\frac{\mathrm{d}^2g}{\mathrm{d}\zeta^2}+(k-\zeta)\frac{\mathrm{d}g}{\mathrm{d}\zeta}-\frac{2}{m}g(\zeta)=0. \tag{29}$$

The complete solution of (29) can be expressed in terms of the confluent hypergeometric function ${}_1F_1(a,b,\zeta)$ which consists of a convergent series for ζ. However, the solution is (see for instance Murphy [16])

$$g(\zeta)=A_1F_1\left(\frac{2}{m},k,\zeta\right)+B\zeta^{1-k}{}_1F_1\left(\frac{2}{m}-k+1,2-k,\zeta\right) \tag{30}$$

where ${}_1F_1(a,b,\zeta)$ is given by the series

$${}_1F_1(a,b,\zeta)=1+\sum_{r=0}^{\infty}\frac{(a)_r\zeta^r}{(b)_r r} \tag{31}$$

and $(a)_r$, $(b)_r$ are Pochhammer's symbols defined by

$$(a)_r=\frac{\Gamma(a+r)}{\Gamma(a)} \qquad (b)_r=\frac{\Gamma(b+r)}{\Gamma(b)}$$

whereas A and B are two arbitrary constants. Inverting the transformations used previously, one can write the most general similarity solution of the integro-differential fragmentation equation with no mass loss. However, the solution is

$$\begin{aligned}u(x,t)=&\left(A_1F_1\left(1+\frac{2}{m},k+1,\frac{m}{a_1^2}(a_1t+a_2)\left(f_0-\frac{a_1}{m}x^m\right)\right)\right.\\&+B\left((a_1t+a_2)\left(f_0-\frac{a_1}{m}x^m\right)\right)^{-k}\\&\left.\times{}_1F_1\left(\frac{2}{m}-k,1-k,\frac{m}{a_1^2}(a_1t+a_2)\left(f_0-\frac{a_1}{m}x^m\right)\right)\right)\\&\times(a_1t+a_2)^k\exp\left[-\left[\frac{mf_0}{a_1}t\right]\right].\end{aligned} \tag{32}$$

In the particular case $a_2=f_0=0$, the general similarity solution (24) reduces to

$$s=tx^m \qquad u=t^kF(s). \tag{33}$$

Inserting (33) into fragmentation equation (4), one obtains the reduced integro–differential equation

$$s\frac{\mathrm{d}F(s)}{\mathrm{d}s}+(1+s)F-\frac{2}{m}\int_s^{\infty}F(s')\,\mathrm{d}s'=0. \tag{34}$$

Differentiating this equation, one obtains the following second order differential equation:

$$s\frac{\mathrm{d}^2F}{\mathrm{d}s^2}+(1+k+s)\frac{\mathrm{d}F}{\mathrm{d}s}+\left(1+\frac{2}{m}\right)F=0.$$

Scaling the similarity variable s by $s=-z$, the above equation can be transformed to Kummer's equation

$$z\frac{\mathrm{d}^2F}{\mathrm{d}z^2}+(1+k-z)\frac{\mathrm{d}F}{\mathrm{d}z}-\left(1+\frac{2}{m}\right)F=0. \tag{35}$$

In terms of the original variables, the fragmentation equation with no mass loss admits the following solution:

$$u(x,t) = t^k \left(A_1F_1\left(1+\frac{2}{m}, k+1, -tx^m\right) + B(tx^m)^{-k}{}_1F_1\left(\frac{2}{m}-k, 1-k, -tx^m\right)\right) \tag{36}$$

which is exactly the same as that obtained previously by Baumann *et al* [10]. In order to obtain the Kummer-type solution which was discussed by McGrady and Ziff [17] and Corngold and Williams [4], we have to impose a further restriction on the similarity solution (33). By setting $k = 1$ in (33) and (36), the solution (36) goes back to that obtained by these authors,

$$u(x,t) = t\left(A_1F_1\left(1+\frac{2}{m}, 2, -tx^m\right) + Bt^{-1}x^{-m}{}_1F_1\left(\frac{2}{m}-1, 0, -tx^m\right)\right). \tag{37}$$

Also, in another paper [20] Ziff generalized the discrete fragmentation equation by introducing a new class of fragmentation model, characterized by

$$a(\tilde{x}) = \tilde{x}^{\alpha} \qquad yk(\tilde{x}, y) = \beta\mu\left(\frac{\tilde{x}}{y}\right)^{\mu-2} + (1-\beta)\delta\left(\frac{\tilde{x}}{y}\right)^{\delta-2} \tag{38}$$

with four adjustable parameters α, β, μ and δ. This class of fragmentation includes, as a special case, our model if $\beta = 1$ and $\eta = 2 - \nu = 2\varphi$. Without performing Lie analysis, Ziff obtained only a scaling solution for the fragmentation model (38) and he made a special ansatz for the solution to obtain Kummer's solution. His solution is equivalent to (37) in the case $\beta = 1$ and $\eta = 2 - \nu = 2\varphi$. Our procedure delivers systematically, in addition to the scaling solution, other classes of similarity solutions. These types of solutions follow from the two cases 1 and 4 that are listed in table 1. In future, using the fragmentation model of Ziff (38) we shall apply Lie's technique to find and classify all possible similarity solutions of the fragmentation equation.

(ii) Case $\varepsilon \neq 0$. Here we extend the above analysis to include the effect of fragmentation mass loss. Based on using the Lie group method, we distinguish six classes of similarity solutions.

Class χ_1. This class corresponds to time translation invariance. As shown in table 2, the similarity variable s and similarity solution $F(s)$ are given by

$$s = x \qquad u = F(s). \tag{39}$$

Substitution of the similarity solution (39) into (4) results in the first order ordinary integro–differential equation

$$(1-a)s^m F(s) - bs^{m+1}\frac{\mathrm{d}F(s)}{\mathrm{d}s} - 2\int_s^{\infty} s'^{m-1}F(s')\,\mathrm{d}s' = 0. \tag{40}$$

Via the substitution

$$g(s) = \int_s^{\infty} s'^{m-1}F(s')\,\mathrm{d}s' \tag{41}$$

equation (40) can be transformed into the second order differential equation

$$bs^2\frac{\mathrm{d}^2g}{\mathrm{d}s^2} - (1-a-b(1-m))s\frac{\mathrm{d}g}{\mathrm{d}s} - 2g = 0 \tag{42}$$

which has the following solution

$$g = c_1s^{r_1} + c_2s^{r_2} \tag{43}$$

where c_1 and c_2 are constants, and r_1 and r_2 are the roots of the quadratic equation

$$br^2 - r(1-a-b(1-m)) - 2 = 0. \tag{44}$$

Making use of the substitution (41), one gets

$$F(s) = -s^{1-m}\frac{\mathrm{d}g}{\mathrm{d}s}. \tag{45}$$

Thus, the solution in terms of the original coordinates reads

$$u = -x^{1-m}(c_1 x^{r_1-1} + c_2 x^{r_2-1}). \tag{46}$$

It is worth noting that in the case of discrete loss of mass (i.e. $\varepsilon = a = b = 0$), equation (44) reduces to $r = -2$ and then the solution (46) goes back to that obtained previously by Baumann *et al* [10],

$$u = c_1 x^{-(m+2)}. \tag{47}$$

Now, we conclude that the obtained stationary solution (47) is thus the similarity solution that corresponds to time translation and scaling invariance of the independent variable u.

Class χ_2. In this case the similarity variable s and similarity solution $F(s)$ are given by

$$s = tx^m \qquad u = F(s). \tag{48}$$

Inserting (48) into fragmentation equation (4), one obtains the reduced integro–differential equation

$$s\frac{\mathrm{d}F}{\mathrm{d}s} + (1-a)sF(s) - mbs^2\frac{\mathrm{d}f}{\mathrm{d}s} - \frac{2}{m}\int_s^\infty F(s')\,\mathrm{d}s' = 0. \tag{49}$$

Differentiating equation (49), one obtains the following second order differential equation

$$s(1-mbs)\frac{\mathrm{d}^2F}{\mathrm{d}s^2} + (1 + (1-a-2mb)s)\frac{\mathrm{d}F}{\mathrm{d}s} + \left(1 - a + \frac{2}{m}\right)F = 0. \tag{50}$$

The solution of this equation can be expressed in terms of hypergeometric function ${}_2F_1(\cdots)$ and MeijerG function [18]. However, the solution in terms of original coordinates is

$$\begin{aligned} u(x,t) = {} & C_1\,{}_2F_1[-k_4(k_1+mk_2), k_4(-k_1+mk_2), 1, bmx^m t] \\ & + C_2\mathrm{MeijeG}[\{\{\}, \{mk_4(k_3+k_2) - mk_4(-k_3+k_2)\}\}, \{\{0,0\},\{\}\}, bmx^m t] \end{aligned} \tag{51}$$

where

$$\begin{aligned} & k_1 = m(1-a-b) \qquad && k_2 = [-4b(-2+(a-1)m) + (a-1+bm)^2]^{1/2} \\ & k_3 = 1 - a + bm && k_4 = \tfrac{1}{2bm^2}. \end{aligned} \tag{52}$$

In the absence of mass loss where $a = b = \varepsilon = 0$, the reduced equation (50) becomes

$$s\frac{\mathrm{d}^2F}{\mathrm{d}s^2} + (1+s)\frac{\mathrm{d}F}{\mathrm{d}s} + \left(1 + \frac{2}{m}\right)F = 0$$

which admits the solution

$$u(x,t) = t\left(A_1F_1\left(1 + \frac{2}{m}, 2, -tx^m\right) + Bt^{-1}x^{-m}{}_1F_1\left(\frac{2}{m} - 1, 0, -tx^m\right)\right) \tag{53}$$

obtained previously by McGrady and Ziff [17] and Corngold and Williams [4]. This leads one to the conclusion that the solution obtained by these authors is thus the similarity solution associated with the subgroup χ_2.

Class χ_3. The similarity variable s and similarity solution $F(s)$ in this case are given by

$$s = t^2x^m - \frac{1}{mb}t \qquad u = t^{\frac{a-1}{mb}}F(s). \tag{54}$$

Substituting (54) into (4), $F(s)$ must satisfy the ordinary integro–differential equation

$$(1-a)sF(s) - \frac{1}{mb}s^2\frac{\mathrm{d}f}{\mathrm{d}s} - \frac{2}{m}\int_s^\infty F(s')\,\mathrm{d}s' = 0. \tag{55}$$

Introducing the transformation

$$g(s) = \int_s^\infty F(s')\,\mathrm{d}s' \tag{56}$$

equation (55) becomes

$$s^2\frac{\mathrm{d}^2 g}{\mathrm{d}s^2} - mb(1-a)s\frac{\mathrm{d}g}{\mathrm{d}s} - 2bg(s) = 0. \tag{57}$$

This equation admits the general solution

$$g = c_1 s^{r_1} + c_2 s^{r_2} \tag{58}$$

where c_1 and c_2 are constants, and r_1 and r_2 are the roots of the quadratic equation

$$r^2 - (mb - mba + 1)r - 2b = 0. \tag{59}$$

By means of (56), $F(s)$ follows

$$F(s) = c_1 s^{r_1-1} + c_2 s^{r_2-1}. \tag{60}$$

In terms of the original coordinates, the similarity solution is as follows:

$$u = t^{\frac{a-1}{mb}}\left[c_1\left(t^2x^m - \frac{t}{mb}\right)^{r_1-1} + c_2\left(t^2x^m - \frac{t}{mb}\right)^{r_2-1}\right]. \tag{61}$$

It is to be noted that this type of solution (61) cannot be constructed from the work of Saied and El-Wakil [13] because the symmetry group of χ_3 was completely missed in their results.

Class $\chi_2 + \gamma\chi_4$. In this case we consider the following linear combination $\chi_2 + \gamma\chi_4$. The corresponding similarity variable s and similarity solution are

$$s = tx^m \qquad u = t^\gamma F(s). \tag{62}$$

The reduced equation is found to be

$$(\gamma + (1-a)s)F(s) + s(1-mbs)\frac{\mathrm{d}F}{\mathrm{d}s} - \frac{2}{m}\int_s^\infty F(s')\,\mathrm{d}s'. \tag{63}$$

Using the substitution

$$g(s) = \int_s^\infty F(s')\,\mathrm{d}s' \tag{64}$$

equation (63) is transformed into the ordinary differential equation

$$s(1-mbs)\frac{\mathrm{d}^2 g}{\mathrm{d}s^2} + (\gamma + (1-a)s)\frac{\mathrm{d}g}{\mathrm{d}s} + \frac{2}{m}g(s) = 0. \tag{65}$$

Solving equation (65) and inverting the transformations used previously, one can write the similarity solution of the fragmentation equation in terms of the original coordinates in the following form:

$$u(x,t) = t^\gamma\Bigg(c_1\,{}_2F_1\left(D_4 - D_5, -\frac{1}{2} - \frac{\mathrm{D}_2}{2bm} + D_5, D_1, bmtx^m\right) + c_2(tx^m)^{1-D_1}\,{}_2F_1(D_6 - D_5, D_6 + D_5, 2 - D_1, bmtx^m)\Bigg) \tag{66}$$

where c_1 and c_2 are constants and

$$D_1 = 1+\gamma \qquad D_2 = 1-a-2mb \qquad D_3 = 1-a+\frac{2}{m}$$
$$D_4 = -\frac{1}{2}-\frac{D_2}{2bm} \qquad D_5 = \frac{4D_3bm+(D_2-bm)^{1/2}}{2bm} \qquad D_6 = \frac{1}{2}-D_1-\frac{D_2}{2bm}. \tag{67}$$

Class $\chi_1 - \gamma\chi_4$. The similarity variable s and similarity solution are

$$s = x \qquad u = e^{-\gamma t}F(s) \tag{68}$$

where the function $F(s)$ must satisfy the equation

$$(-\gamma+(1-a)s^m)F(s) - bs^{m+1}\frac{df}{ds} - 2\int_s^\infty s'^{m+1}F(s)'\,ds' = 0. \tag{69}$$

By the substitution

$$g(s) = \int_s^\infty s'^{m-1}F(s')\,ds' \tag{70}$$

the reduced equation (69) becomes

$$bs^2\frac{d^2g}{ds^2} + s(\gamma s^{-m} - (1-a-b(1-m)))\frac{dg}{ds} - 2g(s) = 0. \tag{71}$$

Solving this equation in connection with (70) and (68), the similarity solution in the original coordinates has the following form:

$$u(x,t) = x^{-m+1}\frac{d}{dx}\left[c_1x^{f_4}\,{}_1F_1\left(f_3-\frac{f_2}{m}, 1-\frac{2f_2}{m}, \frac{\gamma x^{-m}}{bm}\right)\right.$$
$$\left. + c_2x^{f_5}\,{}_1F_1\left(f_3+\frac{f_2}{m}, 1+\frac{2f_2}{m}, \frac{\gamma x^{-m}}{bm}\right)\right]e^{-\gamma t} \tag{72}$$

where

$$f_1 = \frac{a-1-bm}{2b} \qquad f_2 = \frac{(4b+(1-a+bm)^2)^{1/2}}{2b}$$
$$f_3 = -\frac{1}{2}-\frac{1}{2bm}+\frac{a}{2bm} \qquad f_4 = f_1-f_2 \qquad f_5 = f_1+f_2. \tag{73}$$

In the limiting case $\varepsilon = a = b = 0$, equation (71) becomes

$$s(\gamma s^{-m}-1)\frac{dg}{ds} - 2g(s) = 0. \tag{74}$$

Using the transformation $z = s^{-m}$, equation (74) becomes

$$-m(\gamma z^2 - z)\frac{dg}{dz} - 2g(z) = 0. \tag{75}$$

This equation admits the solution

$$g = g_0(\gamma - z^{-1})^{-\frac{2}{m}}. \tag{76}$$

By means of the substitution (64), the function $F(s)$ in terms of s follows

$$F = F_0(\gamma - s^m)^{-\frac{2}{m}-1}. \tag{77}$$

Inverting the transformation used, above, the similarity solution of the fragmentation equation with no mass loss in terms of the original coordinates is

$$u = u_0\,e^{-\gamma t}(\gamma - x^m)^{-\frac{2}{m}-1}. \tag{78}$$

This solution goes back to the time independent solution (47) if $\gamma = 0$.

Class $\chi_1 + \chi_2 + \chi_3 + \chi_4$. This class yields the most general similarity solution of the fragmentation equation (4) with continuous mass loss. As shown in table 2, the general similarity solution and similarity variable are

$$u = F(s) \exp \int_{t_0}^{t} \frac{A(t')}{B(t')} \, dt' \tag{79}$$

where

$$\begin{aligned} s &= \left(\frac{a_0}{2} t^2 + a_1 t + a_2 \right) x^m - m f_0 t \\ A(t) &= m f_0 (a - 1) t + a_3 \\ B(t) &= \frac{a_0}{2} t^2 + a_1 t + a_2. \end{aligned} \tag{80}$$

Inserting equations (79) and (80) into (4) yields, after lengthy manipulations, the following reduced fragmentation equation:

$$(s(s - A_1) + A_2) \frac{d^2 g}{ds^2} - (A_3 s + A_4) \frac{dg}{ds} - A_5 g(s) = 0 \tag{81}$$

where the function $g(s)$ is related to $F(s)$ via

$$g(s) = \int_s^{\infty} F(s') \, ds' \tag{82}$$

and A_i ($i = 1$–5) are given by

$$A_1 = \frac{a_1}{bm} \qquad A_2 = \frac{a_0}{2b^2 m^2} \qquad A_3 = \frac{a_3}{bm} \qquad A_4 = \frac{a_3}{bm} \qquad A_5 = \frac{1}{bm^2}. \tag{83}$$

Solving equation (81) in connection with (82), (79) and (80) yields the general similarity solution of the fragmentation equation with continuous mass loss in terms of the original coordinates u, x, t in the following form:

$$\begin{aligned} u(x,t) = \Bigg\{ & c_1 {}_2F_1 \left[E_1 - E_2, E_1 + E_2, E_3, \frac{B(t)x^m - m f_0 t}{A_1} \right] + c_2 (B(t) x^m - m f_0)^{-A_4/A_1} \\ & \times {}_2F_1 \left[E_4 - E_2, E_4 + E_2, E_5, \frac{B(t)x^m - m f_0 t}{A_1} \right] \Bigg\} \exp \int_{t_0}^{t} \frac{A(t')}{B(t')} \, dt' \end{aligned} \tag{84}$$

where

$$\begin{aligned} E_1 &= \frac{1}{2} - \frac{A_3}{2} \qquad E_2 = \frac{1}{2} \sqrt{1 + 2A_3 + A_3^2 + 4A_5} \qquad E_3 = 1 + -\frac{A_4}{A_1} \\ E_4 &= \frac{1}{2} - \frac{A_3}{2} - \frac{A_4}{A_1} \qquad E_5 = 1 - \frac{A_4}{A_1}. \end{aligned}$$

5. Conclusions

In this paper we discuss a new application of the classical method of Lie group theory to the integro–differential fragmentation equation with continuous mass loss. With the help of the Lie method, the partial integro–differential fragmentation equation is reduced to ordinary integro–differential equations which are solvable explicitly. We obtain a complete classification of all possible non-trivial similarity solutions. These solutions include the effect of both continuous and discrete mass loss and are compared with solutions found earlier [4, 10, 13, 17]. With our calculations based on using Lie group theory for integro–differential equations such as fragmentation equation, we have demonstrated that some of the solutions found previously can be obtained as particular classes from our solutions. Therefore, one may conclude that

Lie's similarity method represents one of the most powerful analytical techniques for solving either differential or integro–differential equations. Another way of finding new classes of similarity solution for the fragmentation equation is the non-classical Lie method introduced by Bluman and Cole [19]. In a future work, we shall deal with such a technique for solving the fragmentation with the model introduced by (38).

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